

Calculus III: Coursework 4
M.A.H. MacCallum 2007

The deadline is 4PM on Tuesday Oct 23rd. Hand-in is to the green box on the ground floor of the Maths building.

1. Using the identities for differentiation of products,

- (a) Evaluate the divergence and curl of $\mathbf{a}(\mathbf{r} \cdot \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors. [You may find it useful to work out $\nabla(\mathbf{b} \cdot \mathbf{r})$ first.]
- (b) Prove that if $\mathbf{A}$ and $\mathbf{B}$ are irrotational, then $\mathbf{A} \times \mathbf{B}$ is solenoidal.
- (c) Prove that for any function $f(r)$, $f(r)\mathbf{r}$ is irrotational.
- (d) Prove that $(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$.

[40]

Answer

(a) $\nabla(\mathbf{r} \cdot \mathbf{b}) = \mathbf{r} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{r}) + (\mathbf{r} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{r} = (\mathbf{b} \cdot \nabla)\mathbf{r} = \mathbf{b}$
using the facts that all derivatives of $\mathbf{a}$ and $\mathbf{b}$ vanish, $\nabla \times \mathbf{r} = \mathbf{0}$, and $(\mathbf{b} \cdot \nabla)\mathbf{r} = \mathbf{b}$. The last two of these were proved in lectures. Hence

$$\nabla \cdot (\mathbf{a}(\mathbf{r} \cdot \mathbf{b})) = \mathbf{a} \cdot \nabla(\mathbf{r} \cdot \mathbf{b}) + (\mathbf{r} \cdot \mathbf{b})(\nabla \cdot \mathbf{a}) = \mathbf{a} \cdot \mathbf{b} \text{ and}$$

$$\nabla \times (\mathbf{a}(\mathbf{r} \cdot \mathbf{b})) = (\mathbf{r} \cdot \mathbf{b})\nabla \times \mathbf{a} - \mathbf{a} \times \nabla(\mathbf{r} \cdot \mathbf{b}) = -\mathbf{a} \times \mathbf{b}$$

$$(b) \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} = 0.$$

(c)

$$\begin{aligned} \nabla \times (f(r)\mathbf{r}) &= f(r)\nabla \times \mathbf{r} - \mathbf{r} \times \nabla f \\ &= -\mathbf{r} \times \nabla f \\ &= -\mathbf{r} \times \frac{\partial f}{\partial r} \nabla r \\ &= -\frac{\partial f}{\partial r}(\mathbf{r} \times \mathbf{r})/r = \mathbf{0} \end{aligned}$$

(d) $\nabla(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = 2(\mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v})$
and rearrange.

Marks: Part (a), 5 for each part plus 5 for $\nabla(\mathbf{r} \cdot \mathbf{b})$ wherever it is calculated (in each bit, 2 for choosing a correct identity, 3 for accuracy). (b) 5, (c) 4 for first identity and accurate use, 3 for chain rule use, 3 for the rest, (d) 4 for using the correct identity, 6 for accurate completion.

2. Evaluate both sides of the Divergence Theorem for the vector field $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + 0\mathbf{k}$ integrated over the box bounded by the planes $x = 0, a, y = 0, b, z = 0, c$ and hence show that the two sides are equal. [23]

Answer

$\nabla \cdot \mathbf{F} = y + z$ so $\int \nabla \cdot \mathbf{F} dV$ over the required region is

$$\begin{aligned} \int_0^c \int_0^b \int_0^a (y + z) dx dy dz &= \int_0^c \int_0^b a(y + z) dy dz \\ &= \int_0^c (a[\frac{1}{2}y^2]_0^b + az[y]_0^b) dz \\ &= \int_0^c (\frac{1}{2}ab^2 + abz) dz \\ &= \frac{1}{2}ab^2[z]_0^c + \frac{1}{2}ab[z^2]_0^c = \frac{1}{2}abc(b + c) \end{aligned}$$

The surface integral has 6 parts, one for each face. On $x = a$, $\mathbf{n} = \mathbf{i}$ and $\mathbf{F} \cdot \mathbf{n} = xy = ay$ so this face contributes

$$\int_0^c \int_0^b ay dy dz = \int_0^c \frac{1}{2}ab^2 dz = \frac{1}{2}ab^2c.$$

On $x = 0$, $\mathbf{F} \cdot \mathbf{n} = 0$, so there is no contribution, and similarly on $y = 0$, $z = 0$, c . We still have $y = b$ to do, where $\mathbf{n} = \mathbf{j}$ and $\mathbf{F} \cdot \mathbf{n} = bz$. This contributes

$$\int_0^c \int_0^a bz dx dz = \frac{1}{2}abc^2.$$

Adding the two non-zero contributions we get the same result as before.

Marks: 4 for the divergence, 6 for volume integration, 3 for knowing how to get correct normals etc on at least one face, 1 each for the four faces giving zero, 3 each for the other two faces.

Note: An example in Thomas integrates xyz over a box. Some students have misinterpreted this as saying the equation for the surface of the box is $xyz = \text{constant}$!

3. By direct evaluation over the surface, calculate

$$\int_S \mathbf{F} \cdot \mathbf{n} dS$$

where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and

(a) S is the rectangular box formed by the planes

$$x = \pm a, \quad y = \pm b, \quad z = \pm c.$$

(b) S is the sphere of radius a , centre the origin.

In both cases, the outward normal $\mathbf{n}$ should be taken.

[25]

Answer

(a) On the surfaces $x = \pm a$ the normals are $\pm\mathbf{i}$ respectively, and $\mathbf{F} \cdot \mathbf{n}$ is $\pm x = a$ on both surfaces so the total for these two surfaces is

$$2 \int_{-b}^b \int_{-c}^c a dz dy = 8abc.$$

The other pairs of faces similarly give the same result twice more and so the total is $24abc$.

(b) On the sphere $\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2} = \mathbf{e}_r = \mathbf{F}/a$ so we get

$$\int_S a dS = 4\pi a^3$$

(using the area of the sphere). There are several correct ways of getting $\mathbf{n}$ e.g. taking the unit vector parallel to $\nabla(x^2 + y^2 + z^2)$.

Marks: (a) correct forming of integrals needed 5 as above, accuracy: 4 for one correct pair of faces, 3 for completion (b) forming correct $\mathbf{n}$ 5, thence the integral 3, accuracy in evaluation 5.

4. Show that over any closed surface $\mathcal{S}$

$$\int_S \mathbf{r} \cdot d\mathbf{S} = 3 \times \text{enclosed volume} .$$

Hence explain the answers to the previous question.

[12]

Answer

By the Divergence Theorem, $\int_S \mathbf{r} \cdot d\mathbf{S} = \int_{\mathcal{D}} \nabla \cdot \mathbf{r} dV$ where $\mathcal{D}$ is the enclosed region. $\nabla \cdot \mathbf{r} = 3$ so this is $\int_{\mathcal{D}} 3dV = 3V$ where V is the volume enclosed.

In the previous question, the two volumes are a rectangular box with sides $2a$, $2b$ and $2c$ and hence of volume $8abc$, and a sphere of radius a and volume $\frac{4}{3}\pi a^3$, while $\mathbf{F}$ is just $\mathbf{r}$. Thus using the divergence theorem the results of the previous question are examples of the calculation done in this question.

Marks: 4 for correct use of Div. Thm., 4 for completing first part, 4 for correct comments on previous question.